3/18/26 Quiz!

Last time:

<u>Theorem</u> (3.3.4)

Let V be a subspace of $\mathbb{R}^n$ with $\boxed{\dim(V) = m}$. Then

(a) We can find <u>at most</u> m lin. indep vectors in V

(b) We need <u>at least</u> m vectors to span V.

(c) If $\boxed{m}$ vectors in V are lin indep then they span V.

(d) If $\boxed{m}$ vectors in V span V then they're lin indep

We proved (a). Parts (b), (d) are in HW.

(c) Assume $\vec{v}_1, \dots, \vec{v}_m$ are lin indep.

$$\boxed{\text{WTS they span } V.}$$

Let $\vec{v} \in V$ be <u>any</u> element.

<u>Claim 1</u> $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$ are <u>linearly dependent</u>

By contradiction. If they are linearly indep then we have a lin. indep set with $m+1$ elements and a spanning set with m elements.

$\Longrightarrow$ $m+1 \leq m$. Contra

By thm 3.3.1

<u>Claim 2</u> $\vec{v}$ is a lin. comb. of $\vec{v}_1, \dots, \vec{v}_m$.

By Claim 1, we have scalars $a_1, \dots, a_m, b$, not all 0, so that

$$a_1\vec{v}_1 + \dots + a_m\vec{v}_m + b\vec{v} = \vec{0}$$

If $b=0$ then $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = 0$ with not all coeff's $=0$, which contradicts lin. indep. So $b \neq 0$. Then

$$\vec{v} = \frac{-a_1}{b}\vec{v}_1 + \dots + \frac{-a_m}{b}\vec{v}_m$$

Thus $\vec{v}_1, \dots, \vec{v}_m$ span (since $\vec{v}$ was arbitrary). //

We have the notion of a basis for <u>any</u> subspace of $\mathbb{R}^n$.

We also have important classes of subspaces:

if $T: \mathbb{R}^m \to \mathbb{R}^n$ is a lin transf. then

- $\ker(T)$ is a subspace of $\mathbb{R}^m$
- $\operatorname{im}(T)$ " " " " $\mathbb{R}^n$

Now we'll develop methods to find bases of these subspaces.

- Basis for $\ker(T)$.

We know $T: \mathbb{R}^m \to \mathbb{R}^n$ is defined by mult. by an $n\times m$ matrix A.

We also know $\ker(T)$ is the solution set of the homog linear system

$$AX = \vec{0}$$ (which we know how to find)

So we'll find our basis via $RREF(A)$

- basis for $\operatorname{im}(T)$.

We know $\operatorname{im}(T)$ is the column space of A.

A bit surprisingly, we'll use $RREF(A)$ also for this.

Let's show the methods, for both $\ker(T)$ and $\operatorname{im}(T)$ following an example in the book.

Ex

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

First find RREF(A). Skipping details, it's

$$B = RREF(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(A has rank 2)

From what we know about row reduction (elem row transformations), the solution of a linear system doesn't change under row operations. So

$$\ker(A) = \ker(B).$$

The solution of $AX = \vec{0}$ is then (solving for the variables corresp. to leading 1's + giving the rest arbitrary values)

$x_1 = -2x_2 - 3x_4 + 4x_5$

$x_2 = s$ (arb.)

$x_3 = 4x_4 - 5x_5$

$x_4 = t$ (arb)

$x_5 = r$ (arb)

So if $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} \in \ker A$ then

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} r$$

So $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$ definitely span $\ker(A)$

Looking at the 1's, these 3 vectors are lin indep:

if $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

then $s = t = r = 0$ so lin indep.

So $\ker(A)$ is 3-dim'l with basis $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

Done $\ker(T) = \ker(A)$